Unbiased simulation of distributions with explicitly known integral transforms

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Abstract In this paper, we propose an importance-sampling based method to obtain an unbiased simulator to evaluate expectations involving random variables whose probability density functions are unknown while their Fourier transforms have an explicit form. We give a general principle about how to choose appropriate importance samplers under different models. Compared with the existing methods, our method avoids time-consuming numerical Fourier inversion and can be applied effectively to high dimensional financial applications such as option pricing and sensitivity estimation under Heston stochastic volatility model, high dimensional affine jump-diffusion model, and various Levy processes.

1 Introduction

Nowadays there are three main techniques for option pricing in finance: Monte Carlo, PDE and Fourier transform methods. The use of Monte Carlo technique in option pricing has a long history going back to Boyle [2] and we refer to Glasserman [7] for an overview. In the case of Lévy-driven models, a basic building block of any Monte-Carlo method is the simulation of the increments of the underlying Lévy process. In some situations - for instance, for the Variance Gamma model - the process can be expressed in terms of subordinated Brownian motion, and hence its increments can be simulated (almost) exactly. However, in other cases no exact simulation algorithm is known. If our aim is to compute some expectation with respect to the distribution

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of the Lévy increments, we can use the so-called Fourier transform approach. Since the seminal work of Carr and Madan [3] Fourier techniques have become well-established in computational finance to efficiently price financial instruments, like European, Asian, multi-asset or barrier options. Recently, Biagini et al. [1] and Hurd and Zhou [8] extend the Fourier method to price options on several assets, considering basket options, spread options and catastrophe insurance derivatives. The Fourier methods belong to the class of numerical integration methods and such they are limited to low dimensional distributions. In this paper we propose a novel approach for computing high-dimensional integrals with respect to distributions with explicitly known Fourier transforms based on the ge! nuine combination of Fourier and Monte Carlo techniques. In order to illustrate the main idea of our approach, let us first consider a simple problem of computing expectations with respect to one-dimensional stable distributions. Let $p_\alpha(x)$ be the density of a random variable X having a symmetric stable law with the stability parameter $\alpha \in (0,2)$, i.e.,

$$\mathscr{F}[p_\alpha](u) = \exp(-|u|^\alpha).$$

Suppose we want to compute the expectation $Q = \mathsf{E}[g(X)]$ for some nonnegative function g. Since there are several algorithms of sampling from stable distribution (see, e.g. [4]), we could use Monte Carlo to construct the estimate

$$Q_n = \frac{1}{n}\sum_{i=1}^{n} g(X_i),$$

where $X_1,\ldots,X_n$ is an i.i.d. sample from the corresponding α-stable distribution. Take, for example, $g(x) = (\max\{x,0\})^\beta$ with some $\beta \in (0,\alpha)$, then we have by the Parseval's identity

$$\begin{aligned}\mathsf{E}[g(X)] &= \int_{-\infty}^{\infty} \frac{g(x)}{x}[x\cdot p_\alpha(x)]\,dx \\ &= \frac{\alpha}{\pi}\int_0^\infty u^{\alpha-1}\exp(-u^\alpha)\,\mathsf{Im}\,[\mathscr{F}[g(\cdot)/\cdot](u)]\,du, \\ &= \frac{\alpha\,\Gamma(\beta)\sin(\beta\pi/2)}{\pi}\int_0^\infty u^{\alpha-\beta-1}\exp(-u^\alpha)\,du\end{aligned}$$

provided $\alpha > 1$. This means that $\mathsf{E}[g(X)] = \mathsf{E}[g'(X')]$, where X' has a power exponential distribution with the density

$$f_\alpha(x) = \frac{1}{\Gamma(1+1/\alpha)}\exp(-x^\alpha)$$

and $g'(x) = C(\alpha,\beta)|x|^{\alpha-\beta-1}$ with $C(\alpha,\beta) = \frac{\Gamma(1/\alpha)\Gamma(\beta)\sin(\beta\pi/2)}{\pi}$. In particular, if $\beta = \alpha - 1$, then $\mathsf{Var}[g(X)] > B(2-\alpha)^{-1}$ for some $B > 0$ not depending on α, while $\mathsf{Var}[g'(X')] = 0$. This shows that even in the above very simple situation, moving to the Fourier domain can significantly reduce the variance of Monte Carlo estimates. More importantly, by using our approach, we replace the problem of sampling from the stable distribution p_α by a much simpler problem of drawing from the exponential power distribution f_α. Of course, the main power of Monte Carlo methods can be observed in high-dimensional integration problems, which will be considered in the next section.

2 General framework

Let g be a real-valued function on $\mathbb{R}^d$ and let p be a probability density on $\mathbb{R}^d$. Our aim is to compute the integral of g with respect to p :

$$V = \int_{\mathbb{R}^d} g(x)p(x)\,dx.$$

Suppose that there is a vector $R \in \mathbb{R}^d$, such that

$$g(x)e^{-\langle x,R\rangle} \in L^1(\mathbb{R}^d), \quad f(x)e^{\langle x,R\rangle} \in L^1(\mathbb{R}^d),$$

then we have by the Parseval's formula

$$V = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathscr{F}[g](iR-u)\mathscr{F}[p](u-iR)\,du. \tag{1}$$

Let q be a probability density function with the property that $q(x) = 0$ whenever $|\mathscr{F}[p](u-iR)| = 0$. That is, q has the same support as $|\mathscr{F}[p](u-iR)|$. Then we can write

$$V = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathscr{F}[g](iR-u)\frac{\mathscr{F}[p](u-iR)}{q(u)}q(u)\,du = \mathsf{E}_q\left[h(X)\right], \tag{2}$$

where

$$h(x) = \frac{1}{(2\pi)^d}\mathscr{F}[g](iR-x)\frac{\mathscr{F}[p](x-iR)}{q(x)}.$$

and X is a random variable distributed according to q. The variance of the corresponding Monte Carlo estimator is given by

$$\mathsf{Var}_q[h(X)] = \left(\frac{1}{2\pi}\right)^{2d}\int_{\mathbb{R}^d} |\mathscr{F}[g](iR-u)|^2\frac{|\mathscr{F}[p](u-iR)|^2}{q(u)}\,du - V^2.$$

Note that the function $|\mathscr{F}[p](iR-u)|$ is, up to a constant, a probability density and in order to minimize the variance, we need to find a density q, that minimizes the ratio $\frac{|\mathscr{F}[p](u-iR)|}{q(u)}$ and that we are able to simulate from. In the next section, we discuss how to get a tight upper bound for $|\mathscr{F}[p](iR-u)|$ in the case of an infinitely divisible distribution p, corresponding to the marginal distributions of Lévy processes. Such a bound can be then used to find a density q leading to small values of variance $\mathsf{Var}_q[h(X)]$.

3 Lévy processes

Let (Z_t) be a pure jump d-dimensional Levy process with the characteristic exponent ψ, that is

$$\mathrm{E}\left[e^{i\langle u,Z_t\rangle}\right] = e^{-t\psi(u)}, \quad u\in\mathbb{R}^d.$$

Consider the process $X_t = \Lambda Z_t$, where Λ is a real $m\times d$ matrix. Let a vector $R\in\mathbb{R}^m$ be such that $\nu_R(dz) \doteq e^{\langle\Lambda^* R,z\rangle}\nu(dz)$ is again a Lévy measure, i.e.

$$\int \left(|z|^2\wedge 1\right)\nu_R(dz) < \infty.$$

Suppose that there exist a constant $C_\nu > 0$ and a real number $\alpha\in(0,2)$, such that, for sufficiently small $\rho>0$, the following estimate holds

$$\int_{\{z\in\mathbb{R}:|\langle z,h\rangle|\le\rho\}} \langle z,h\rangle^2\,\nu_R(dz) \ge C_\nu\rho^{2-\alpha}, \quad h\in\mathbb{R}^d, \quad |h|=1. \tag{3}$$

Lemma 1. *Suppose that* (3) *holds, then there exists constant $A_R>0$ such that, for any $u\in\mathbb{R}^m$ and sufficiently large $|\Lambda^* u|$,*

$$|\phi_t(u-iR)| \le A_R \exp\left(-\frac{2tC_\nu}{\pi^2}|\Lambda^* u|^\alpha\right),$$

where $\phi_t(z)\doteq \mathrm{E}\left[e^{i\langle z,X_t\rangle}\right]$.

Proof. For any $u\in\mathbb{R}^m$, we have

$$\begin{aligned}
|\phi_t(u-iR)| &= \exp\left(-t\int_{\mathbb{R}^d}\left[1-e^{\langle\Lambda^*R,z\rangle}\cos\left(\langle\Lambda^*u,z\rangle\right)+\langle\Lambda^*R,z\rangle\mathbb{1}_{\{|z|\le1\}}\right]\nu(dz)\right)\\
&= \exp\left(-t\int_{\mathbb{R}^d}\left[1-e^{\langle\Lambda^*R,z\rangle}+\langle\Lambda^*R,z\rangle\mathbb{1}_{\{|z|\le1\}}\right]\nu(dz)\right)\\
&\quad\times\exp\left(-t\int_{\mathbb{R}^d}\left[e^{\langle\Lambda^*R,z\rangle}\left\{1-\cos\left(\langle\Lambda^*u,z\rangle\right)\right\}\right]\nu(dz)\right)\\
&= A_R\exp\left(-t\int_{\mathbb{R}^d}\left\{1-\cos\left(\langle\Lambda^*u,z\rangle\right)\right\}\nu_R(dz)\right),
\end{aligned}$$

where

$$A_R = \exp\left(t\int_{\mathbb{R}^d}\left(e^{\langle\Lambda^*R,z\rangle}-1-\langle\Lambda^*R,z\rangle\mathbb{1}_{\{|z|\le1\}}\right)\nu(dz)\right) < \infty,$$

since

$$\left|e^{\langle \Lambda^* R, z\rangle} - 1 - \langle \Lambda^* R, z\rangle \mathbb{1}_{\{|z|\le 1\}}\right| \le C_1(\Lambda^* R)\,|z|^2\,\mathbb{1}_{\{|z|\le 1\}} + C_2(\Lambda^* R)e^{\langle \Lambda^* R, z\rangle}\mathbb{1}_{\{|z|>1\}}.$$

First, note that the condition (3) is equivalent to the following one

$$\int_{\{z\in\mathbb{R}:|\langle z,k\rangle|\le 1\}} \langle z,k\rangle^2\,\nu_R(dz) \ge C_\nu |k|^\alpha,$$

for sufficiently large $k\in\mathbb{R}^d$, say $|k|\ge c_0$. To see this, it is enough to change in (3) the vector h to the vector ρk. Fix $u\in\mathbb{R}^m$ with $|u|\ge 1$ and $|\Lambda^* u|\ge c_0$, then using the inequality $1-\cos(x)\ge \frac{2}{\pi^2}|x|^2$, $|x|\le\pi$, we find

$$\begin{aligned}\int_{\mathbb{R}^d}(1-\cos(\langle \Lambda^* u, z\rangle))\,\nu_R(dz) &\ge \frac{2}{\pi^2}\int_{\{z\in\mathbb{R}:|\langle \Lambda^* u,z\rangle|\le 1\}}\langle \Lambda^* u, z\rangle^2\,\nu_R(dz)\\ &\ge \frac{2C_\nu}{\pi^2}\,|\Lambda^* u|^\alpha.\end{aligned}$$

Discussion

Let us comment on the condition (3) (for simplicity we take $R=0$). Clearly, if (Z_t) is a d-dimensional α-stable process which is rotation invariant ($\psi(h)=c_\alpha|h|^\alpha$, for $h\in\mathbb{R}^d$), then (3) holds. Consider now general α-stable processes. It is known that Z is α-stable if and only if its components $Z^1,\ldots,Z^d$ are α-stable and if the Levy $\mathscr{C}$ copula of Z is homogeneous of order 1, i.e.

$$\mathscr{C}(r\cdot\xi_1,\ldots,r\cdot\xi_d) = r\mathscr{C}(\xi_1,\ldots,\xi_d)$$

for all $\xi=(\xi_1,\ldots,\xi_d)\in\mathbb{R}^d$ and $r>0$. As an example of such homogeneous Levy copula one can consider

$$\mathscr{C}(\xi_1,\ldots,\xi_d) = 2^{2-d}\left(\sum_{j=1}^d |\xi_j|^{-\theta}\right)^{-1/\theta}\left(\eta\mathbb{1}_{\xi_1\cdot\ldots\cdot\xi_d\ge 0} - (1-\eta)\mathbb{1}_{\xi_1\cdot\ldots\cdot\xi_d<0}\right),$$

where $\theta>0$ and $\eta\in[0,1]$. If the marginal tail integrals given by

$$\Pi_j(x_j) = \nu(\mathbb{R},\ldots,\mathscr{I}(x_j),\ldots\mathbb{R})\,\mathrm{sgn}(x_j)$$

with

$$\mathscr{I}(x) = \begin{cases}(x,\infty), & x\ge 0,\\ (-\infty,x], & x<0,\end{cases}$$

are absolutely continuous, we can compute the Lévy measure ν for the Lévy copula $\mathscr{C}$ by differentiation as follows:

$$\nu(dx_1,\ldots,dx_d) = \partial_1\ldots\partial_d\mathscr{C}|_{\xi_1=\Pi_1(x_1),\ldots,\xi_d=\Pi_d(x_d)}\,\nu_1(dx_1)\cdot\ldots\cdot\nu_d(dx_d),$$

where $\nu_1(dx_1),\ldots,\nu_d(dx_d)$ are the marginal Lévy measures. Suppose that the marginal Lévy measures are absolutely continuous with a stable-like behaviour:

$$\nu_j(dx_j) = k_j(x_j)\,dx_j = \frac{l_j(|x_j|)}{|x_j|^{1+\alpha}}\,dx_j, \quad j=1,\ldots,d,$$

where $l_1,\ldots,l_d$ are some nonnegative bounded nonincreasing functions on $[0,\infty)$ with $l_j(0)>0$ and $\alpha\in[0,2]$. Then

$$\nu(dx_1,\ldots,dx_d) = G(\Pi_1(x_1),\ldots,\Pi_d(x_d))\,k_1(x_1)\cdot\ldots\cdot k_d(x_d)\,dx_1\ldots dx_d$$

with $G(\xi_1,\ldots,\xi_d) = \partial_1\ldots\partial_d\mathscr{C}|_{\xi_1\ldots,\xi_d}$. Note that for any $r>0$,

$$k_j(rx_j) = r^{-1-\alpha}\bar{k}_j(x_j,r), \quad \Pi_j(rx_j) = r^{-\alpha}\overline{\Pi}_j(x_j,r), \quad j=1,\ldots,d,$$

where

$$\bar{k}_j(x_j,r) = \frac{l_j(rx_j)}{|x_j|^{1+\alpha}}, \quad \overline{\Pi}_j(x_j,r) = \mathbb{1}_{\{x_j\geq 0\}}\int_{x_j}^{\infty}\bar{k}_j(s,r)\,ds + \mathbb{1}_{\{x_j<0\}}\int_{-\infty}^{x_j}\bar{k}_j(s,r)\,ds.$$

Since the function G is homogeneous with order $1-d$, we get for $\rho\in(0,1)$,

$$\begin{aligned}\int_{\{z\in\mathbb{R}:|\langle z,h\rangle|\leq\rho\}}\langle z,h\rangle^2\,\nu(dz) &= \rho^{2-\alpha}\int_{\{z\in\mathbb{R}:|\langle y,h\rangle|\leq 1\}}\langle y,h\rangle^2 G\left(\overline{\Pi}_1(y_1,\rho),\ldots,\overline{\Pi}_d(y_d,\rho)\right)\\ &\quad \bar{k}_1(y_1,\rho)\cdot\ldots\cdot\bar{k}_d(y_d,\rho)\,dy_1\ldots dy_d\\ &\geq \rho^{2-\alpha}\int_{\{z\in\mathbb{R}:|\langle y,h\rangle|\leq 1\}}\langle y,h\rangle^2 G\left(\overline{\Pi}_1(y_1,1),\ldots,\overline{\Pi}_d(y_d,1)\right)\\ &\quad \bar{k}_1(y_1,1)\cdot\ldots\cdot\bar{k}_d(y_d,1)\,dy_1\ldots dy_d\end{aligned}$$

and the condition (3) holds, provided

$$\inf_{h:|h|=1}\int_{\{z\in\mathbb{R}:|\langle z,h\rangle|\leq 1\}}\langle z,h\rangle^2\,\nu(dz) > 0.$$

If for some $R=(R_1,\ldots,R_d)$ the fucntions $e^{xR_i}l_i(x)$, $i=1,\ldots,d$, are bounded, the condition (3) holds for $\nu_R(dz) \doteq e^{\langle R,z\rangle}\nu(dz)$.

4 Positive definite densities

Let p be a probability density on $\mathbb{R}^d$, which is positive definite. For example, all symmetric infinite divisible absolute continuous distributions have positive definite densities. Let furthermore g be a nonnegative integrable function on $\mathbb{R}^d$. Suppose that we want to compute the expectation

$$V = \mathrm{E}_p[g(X)] = \int_{\mathbb{R}^d} g(x)p(x)\,dx.$$

We have by the Parseval's identity

$$V = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \overline{\mathscr{F}[g](x)}\mathscr{F}[p](x)\,dx.$$

Note that $p^*(x) = \mathscr{F}[p](x)/((2\pi)^d p(0))$ is a probability density and therefore we have another "dual" representation for V : (Remark: Dividing by $(2\pi)^d$) makes p^* a density)

$$V = \mathrm{E}_{p*}[g^*(X)]$$

with $g^*(x) = p(0)\overline{\mathscr{F}[g](x)}$. Let us compare the variances of the random variables $g(X)$ under $X \sim p$ and $g^*(X)$ under $X \sim p^*$. It holds

$$\mathrm{Var}_p[g(X)] = \int_{\mathbb{R}^d} g^2(x)p(x)\,dx - V^2$$

and

$$\begin{aligned}\mathrm{Var}_{p*}[g^*(X)] &= \frac{p(0)}{(2\pi)^d}\int_{\mathbb{R}^d} |\mathscr{F}[g](x)|^2\,\mathscr{F}[p](x)\,dx - V^2\\ &= p(0)\int_{\mathbb{R}^d}(g\star g)(x)p(x)\,dx - V^2,\end{aligned}$$

where

$$(g\star g)(x) = \int g(x+y)g(y)\,dy.$$

As a result,

$$\mathrm{Var}_p[g(X)] - \mathrm{Var}_{p*}[g^*(X)] = \int_{\mathbb{R}^d}\left[g^2(x) - p(0)(g\star g)(x)\right]p(x)\,dx.$$

Note that if $p(0) > 0$ is small, then it is likely that $\mathrm{Var}_p[g(X)] > \mathrm{Var}_{p*}[g^*(X)]$. This means that estimating V under p^* with Monte Carlo can be viewed as a variance reduction method in this case. Apart from the variance reduction effect, the density p^* may have in many cases (for example, for infinitely divisible distributions) much simpler form than p and therefore is easy to simulate from.

5 Numerical examples

European Call option under CGMY model

The CGMY process (X_t) with drift μ is a pure jump Lévy with the Lévy measure

$$\nu_{\mathrm{CGMY}}(x) = C\left[\frac{\exp(Gx)}{|x|^{1+Y}}\mathbb{1}_{x<0} + \frac{\exp(-Mx)}{x^{1+Y}}\mathbb{1}_{x>0}\right],\ C,G,M>0,\ 0<Y<2.$$

As can be easily seen, the Lévy measure ν_{CGMY} satisfies the condition (3) with $\alpha = Y$. The characteristic function of X_T is given by

$$\phi(u) = \mathsf{E}[e^{iuX_T}] = \exp\left\{i\mu uT + TC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]\right\},$$

where $\mu = r - C\Gamma(-Y)[(M-1)^Y - M^Y + (G+1)^Y - G^Y$ ensures that $(e^{-rT}e^{X_T})$ is a martingale. Suppose the stock price follows the model

$$S_t = S_0 e^{X_t},$$

then due to (1), for any $R>1$, the price of the European call option is given by

$$e^{-rT}\mathsf{E}[(S_T-K)^+] = \frac{e^{-rT}}{2\pi}\int \mathscr{F}[g](iR-u)\mathscr{F}[p](u-iR)du, \tag{4}$$

where

$$\mathscr{F}[g](iR-u) = \frac{K^{1-R}e^{-iu\ln K}}{(iu+R-1)(iu+R)},\ \mathscr{F}[p](u-iR) = e^{i(u-iR)\ln S_0}\cdot \mathsf{E}[e^{i(u-iR)X_T}].$$

(Remark: Sorry the original expression is its conjugate.)
Lemma 1 implies

$$|\mathscr{F}[p](u-iR)| \le Ae^{-\frac{|u|^\alpha}{\theta}}$$

for $\alpha \le Y$, some $A>0$, $\theta>0$ and large enough u. So we can use the exponential power density

$$q(u) = \frac{1}{2\theta^{\frac{1}{\alpha}}\Gamma(1+\frac{1}{\alpha})}e^{-\frac{|u|^\alpha}{\theta}}$$

as the important sampler in (2), where the parameter θ can be chosen by minimizing the simulated second moment. Feng, [6] used the parameters $C=1,\ G=5,\ M=5,\ Y=0.5$ to calculate the price of the European call option via numerical inversion of the corresponding characteristic function. The obtained option price was 19.8129. Our numerical results are shown in Table 1.

Table 1 Call option in CGMY model, exponential power IS ($R = 4.5, \alpha = 0.49, \theta = 0.4$)

No. of simulation	Price	95%-interval	RMSE	Time(s)
100,000	19.8156	[19.7793,19.8520]	0.0086	0.08
400,000	19.8109	[19.7927,19.8292]	0.0093	0.29
1,600,000	19.8127	[19.8036,19.8218]	0.0046	1.19

European Put option under NIG model

The Normal Inverse Gaussian Lévy process can be constructed by subordinating BM with an Inverse Gaussian process:

$$X_t(\alpha,\beta,\delta) = \beta T_t(\nu,\delta) + W(T_t(\nu,\delta)),$$

where $\alpha = \sqrt{\beta^2+\nu^2}$ and $T_t(\nu,\delta)$ is the Inverse Gaussian Lévy process defined by $T_t(\nu,\delta) = \inf\{s > 0 : \nu s + B_s = \delta t\}$. We have

$$\mathsf{E}[e^{iuX_t}] = \exp\left(\delta t\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right]\right)$$

and the corresponding Lévy measure ν_{NIG} fulfils the condition (3) with $\alpha = 1$. Suppose the stock price is modelled by

$$S_t = S_0 e^{at+X_t},$$

where the choice $a = r - q - \delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2})$ ensures the martingale condition. Then for any $R < 0$, the price of European put option is given by

$$e^{-rT}\mathsf{E}[(K-S_T)^+] = \frac{e^{-rT}}{2\pi}\int \mathscr{F}[g](iR-u)\mathscr{F}[p](u-iR)du,$$

where

$$\mathscr{F}[g](iR-u) = \frac{K^{1-R}e^{-iu\ln K}}{(iu+R-1)(iu+R)},\ \mathscr{F}[p](u-iR) = e^{i(u-iR)(\ln S_0+aT)}\cdot\mathsf{E}[e^{i(u-iR)X_T}].$$

Lemma 1 implies that one can use the Laplace density

$$q(u) = \frac{1}{2\theta}e^{-\frac{|u|}{\theta}}$$

as the important sampler, where the parameter θ can be chosen by minimizing the simulated second moment. Chen, Feng and Lin [5] used parameters $\alpha = 15,\ \beta = -5,\ \delta = 0.5,\ r = 0.03,\ S_0 = K = 100,\ T = 0.5$ to calculate the price of the European put option and obtained the value 4.5898. Table 2 shows numerical results with the same parameters and compares the RMSE and the computational time of our method with that of the method direct simulating the subordinator.

Table 2 Put option in NIG model, Exponential power IS ($R = 9.3,\ \theta = 2.4$)

No. of simulation	Price	95%-interval	RMSE	Time	RMSE (Direct)	Time
100,000	4.5900	[4.5879,4.5922]	0.0011	0.06	0.0238	0.04
400,000	4.5896	[4.5886,4.5907]	0.0006	0.23	0.0119	0.13
1,600,000	4.5897	[4.5891,4.5902]	0.0003	0.92	0.0059	1.19

Barrier option under CGMY model

The payoff function of barrier option with m monitoring time points is

$$(S_T - K)^+ \mathbb{1}_{\{L \le S_{t_1} \dots S_{t_m} \le U\}},$$

where $S_t = S_0 e^{X_t}$ and $0 < t_1 < \dots < t_m < T$. According to the Parseval's Theorem, the option price is equal to

$$\frac{e^{-rT}}{(2\pi)^{m+1}} \int_{\mathscr{R}^{m+1}} \mathscr{F}[g](i\boldsymbol{R} - \boldsymbol{u}) \mathscr{F}[p](\boldsymbol{u} - i\boldsymbol{R}) d\boldsymbol{u},$$

where $\boldsymbol{u} = (u_1, \dots, u_{m+1}),\ \boldsymbol{R} = (0, \dots, 0, R)$,

$$\begin{aligned}\mathscr{F}[g](i\boldsymbol{R} - \boldsymbol{u}) = & \frac{e^{(-iu_{m+1} - R + 1)\ln K}}{(iu_{m+1} + R - 1)(iu_{m+1} + R)} \frac{e^{(iu_{m+1} - iu_m + R)\ln U} - e^{(iu_{m+1} - iu_m + R)\ln L}}{iu_{m+1} - iu_m + R} \\ & \frac{e^{(iu_m - iu_{m-1})\ln U} - e^{(iu_m - iu_{m-1})\ln L}}{iu_m - iu_{m-1}} \cdots \frac{e^{(iu_2 - iu_1)\ln U} - e^{(iu_2 - iu_1)\ln L}}{iu_2 - iu_2}\end{aligned}$$

and $\mathscr{F}[p](\boldsymbol{u} - i\boldsymbol{R}) = e^{iu_1 \ln S_0} \big[\prod_{j=1}^{m} \phi_\Delta(u_j)\big] \phi_\Delta(u_{m+1} - iR)$.
We use parameters $C = 1,\ G = 5,\ M = 5,\ Y = 1.5,\ r = 0.1,\ s = 100,\ K = 100,\ T = 2,\ U = 105,\ L = 95$ and calculate the price of the barrier option when there is only one monitoring time at $t = 1$. The benchmark price calculated by numerical integration is 9.5880. We use the method described in Section 2 with the importance sampler of the form

$$h(u_1, u_2) = \frac{1}{2\theta_1^{1/\alpha_1} \Gamma(1 + \frac{1}{\alpha_1})} e^{-\frac{|u_1|^{\alpha_1}}{\theta_1}} \frac{1}{2\theta_2^{1/\alpha_2} \Gamma(1 + \frac{1}{\alpha_2})} e^{-\frac{|u_2|^{\alpha_2}}{\theta_2}}$$

where $\alpha_1 = 1.5,\ \theta_1 = 0.9,\ \alpha_2 = 1.5,\ \theta_2 = 0.25$ and the damping factor $R = 1.06$. The results are presented in Table 3.

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Table 3 Barrier option in CGMY model, exponential power IS ($\alpha_1 = 1.5,\ \theta_1 = 0.9,\ \alpha_2 = 1.5,\ \theta_2 = 0.25,\ R = 1.06$)

No. of simulation	Price	95%-interval	RMSE	Time(s)
6,400,000	2.3113	[2.1012,2.5214]	0.1072	9.13
25,600,000	2.3399	[2.2349,2.4450]	0.0536	36.65

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